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Simplified proof of the blocking theorem for free-choice Petri nets

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ABSTRACT

Every cluster in a bounded and live free-choice system has a unique blocking marking. It can be reached by firing an occurrence sequence, which avoids any transition of the cluster. This theorem is due to Gaujal, Haar and Mairesse. We will give a short proof using standard results on CP-subnets of well-formed free-choice nets.

0. Introduction

Blocking markings of a Petri net are reachable markings which enable transitions from only a single cluster. Therefore one obtains a dead marking after removing from the cluster all tokens of the blocking marking. In a free-choice system either all transitions of a given cluster are enabled or no transition is enabled. The blocking theorem states that a bounded and live free-choice system has blocking markings for every given cluster. Such blocking markings are uniquely determined by the cluster. In addition, for every reachable marking there exists an enabled occurrence sequence without transitions from the cluster, such that firing the occurrence sequence leads to the blocking marking.

Historically, the first proposition about blocking markings has been proved for safe and live T-systems by Genrich and Thiagarajan [3]. Later this result has been generalized considerably to bounded and live free-choice systems by Gaujal, Haar and Mairesse [2]. Their proof of the blocking theorem goes by induction on the number of T-components from a covering of the net and uses CP-subnets. The proof employs subtle arguments about occurrence sequences; in particular it uses reverse firing.

In the present paper we will give a shorter proof without reverse firing. Our proof rests on the following ingredients: The equivalence of liveness and deadlock-freeness for bounded, strongly-connected free-choice systems, the reduction of well-formed free-choice nets via CP-subnets and the uniqueness of blocking markings in bounded and live T-systems and certain marked CP-subnets.

1. Prerequisites

We will assume that the reader is familiar with the basic properties of ordinary Petri nets. For the convenience of the reader and to fix the notation we recall some concepts which are used throughout the paper. We consider finite ordinary Petri nets (N, μ_0). Here the net $N = (P, T, F)$ comprises a finite set P of places, a disjoint finite set T of transitions and a set $F \subseteq (P \times T) \cup (T \times P)$ of directed arcs, while $\mu_0 : P \to \mathbf{N}$ denotes the initial marking of the net. We will often dispense with an explicit notation for the set of places, transitions and arcs; we use the shorthand $x \in N$ for a node $x \in P \cup T$. We shall write $pre(x) := \{y \in N\colon\ (y, x) \in F\}$ for the *pre-set* and $post(x) = \{y \in N\colon\ (x, y) \in F\}$ for the *post-set* of a node $x \in N$. If

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$\mu(p) = n \in \mathbf{N}$ for a marking μ of N and a place $p \in P$, we say that p *is marked with* n tokens at μ, and the number n is called the *token content* of p at μ.

A net $N = (P, T, F)$ is *connected*, if every two nodes $x_1, x_2 \in P \cup T$ are joined by a sequence of arcs, i.e. $(x_1, x_2) \in (F \cup F^{-1})^*$. If even $(x_1, x_2) \in F^*$ and $(x_2, x_1) \in F^*$ for every two nodes $x_1, x_2 \in P \cup T$, the net is *strongly-connected*. A *path* in $N = (P, T, F)$ is a sequence $(x_0, x_1, \ldots, x_n)$ with nodes $x_i \in P \cup T$, $0 \leqslant i \leqslant n$, and $(x_i, x_{i+1}) \in F$, $0 \leqslant i < n$. The path is *elementary*, if $x_i \neq x_j$ for all $i \neq j$.

A net N is a *T-net* if all places have exactly one input and exactly one output transition, i.e.

$$card\big[pre(p)\big] = 1 = card\big[post(p)\big]$$

for all places $p \in N$. A *T-system* is a Petri net (N, μ_0) with N a T-net.

A net N is a *free-choice net* if for every two transitions $t_1, t_2 \in N$

$$\text{either} \quad pre(t_1) \cap pre(t_2) = \emptyset \quad \text{or} \quad pre(t_1) = pre(t_2).$$

A marked free-choice net (N, μ_0) is called *free-choice system* or *free-choice Petri net*.

A *subnet* $N' = (P', T', F')$ of a net $N = (P, T, F)$, denoted $N' \subseteq N$, is a net with

$$P' \subseteq P,\ T' \subseteq T,\ F' = F \cap \big[(P' \times T') \cup (T' \times P')\big].$$

If $X \subseteq P \cup T$ is a set of nodes from a net $N = (P, T, F)$, then the triple $(X \cap P, X \cap T, F \cap (X \times X))$ is a subnet of N, called the subnet of N *generated by* X. A subnet $N' \subseteq N$ is called *T-subnet* if N' is a T-net. A T-subnet $N_T \subseteq N$, which is generated by a nonempty subset X of nodes, is a *T-component* of N iff N_T is strongly connected and

$$pre(t) \cup post(t) \subseteq X \quad \text{for all transitions } t \in X.$$

The *complement* of a subnet $N' \subseteq N$ is the subnet $\bar{N} \subseteq N$ generated by $(P - P') \cup (T - T')$. We write $N - N' := \bar{N}$ for short. A subnet $N' \subseteq N$ is called *transition-bordered*, if only transitions of N' are adjacent to nodes from the complement $\bar{N}$, i.e. if any node $x \in P' \cup T'$ satisfying

$$\big[pre(x) \cup post(x)\big] \cap (\bar{P} \cup \bar{T}) \neq \emptyset$$

is a transition. For a transition-bordered subnet $N' \subseteq N$ a transition $t_{in} \in N'$ with $pre(t_{in}) \cap \bar{N} \neq \emptyset$ is called *way-in transition*.

Clusters group conflicting transitions of a net $N = (P, T, F)$ together with their pre-set: The *cluster* of a node $x \in P \cup T$, denoted $cl(x)$, is the minimal set of nodes so that

- $x \in cl(x)$,
- if $p \in P$ belongs to $cl(x)$, then also $post(p) \subseteq cl(x)$, and
- if $t \in T$ belongs to $cl(x)$, then also $pre(t) \subseteq cl(x)$.

An *S-invariant* of a net $N = (P, T, F)$ is a function $\lambda : P \to \mathbf{Z}$ with

$$\sum_{p \in pre(t)} \lambda(p) = \sum_{p \in post(t)} \lambda(p)$$

for all transitions $t \in N$. For an S-invariant λ and a marking μ of N the *scalar product* is defined as

$$\langle \lambda, \mu \rangle := \sum_{p \in P} \lambda(p) \cdot \mu(p).$$

Two markings μ_1 and μ_2 are said *to agree* on an S-invariant λ, iff $\langle \lambda, \mu_1 \rangle = \langle \lambda, \mu_2 \rangle$.

For a net N the *firing rule* defines the firing of a transition: A transition $t \in T$ is *enabled* at a marking μ of N iff each place from $pre(t)$ is marked at μ with at least one token. Being enabled, t may *occur* or *fire*. Firing t yields a new marking μ', which results from μ by consuming one token from each pre-place of t and by creating one additional token on each post-place of t; this is denoted by $\mu \xrightarrow{t} \mu'$. An *occurrence sequence* from μ is a finite sequence $\sigma = t_1 \ldots t_k, k \in \mathbf{N}$, such that $\mu \xrightarrow{t_i} \mu_1, \ldots, \mu_{k-1} \xrightarrow{t_k} \mu_k$. We denote by $\mu \xrightarrow{\sigma} \mu_k$ the fact, that firing σ yields the marking μ_k. A *reachable marking* of a Petri net (N, μ_0) is a marking, which results from firing an occurrence sequence from μ_0. Note that all occurrence sequences in this paper will be considered as finite. A *home state of* (N, μ_0) is a marking μ_{home}, which is reachable from every reachable marking of (N, μ_0), i.e. μ_{home} is a reachable marking of (N, μ) for every reachable marking μ of (N, μ_0).

A Petri net (N, μ_0) is *live* iff for each reachable marking μ and for each transition $t \in T$ the Petri net (N, μ) has a reachable marking which enables t. A Petri net is *bounded* iff there exists a natural number, which bounds from above the token content of every place at every reachable marking. A net N is *well-formed* iff there exists a marking μ_0 of N such that the Petri net (N, μ_0) is live and bounded.

Definition 1.1 *(CP-subnet).* A nonempty, connected and transition-bordered T-subnet $\hat{N}$ of a net N is a *CP-subnet*, if the complement $N - \hat{N}$ is nonempty and strongly-connected.

It is well-known that a CP-subnet of a well-formed free-choice net has a unique way-in transition, cf. [1, Prop. 7.10]. The following Lemma 1.2 is a mild intensification of another well-known result. It will be used in the proof of Theorem 2.4.

Lemma 1.2 *(Existence of CP-subnets). Consider a well-formed free-choice net N, which is not a T-net, and a transition $t \in N$. Then there exists a CP-subnet $\hat{N} \subset N$ with $t \in N - \hat{N}$.*

Proof. We choose a finite covering of N by T-components $(N_{T,i})_{i \in I}$ which is minimal, i.e. no proper subset $J \subset I$ defines a covering $(N_{T,j})_{j \in J}$ of N. Because N is not a T-net, we have $card(I) \geqslant 2$. Attached to the covering we consider the undirected graph G with vertices all indices $i \in I$ and edges $\{i, j\}$, $i \neq j$, iff $N_{T,i} \cap N_{T,j} \neq \emptyset$. Connectedness of N implies, that also G is connected. Any undirected spanning tree of G has at least two leaves $i \neq j \in I$, i.e. vertices with exactly one adjacent edge. They correspond to the T-components $N_{T,i}$ and $N_{T,j}$. The distinguished transition t is either contained in both components, i.e. $t \in N_{T,i} \cap N_{T,j}$, or it is not contained in one component at least, say $t \notin N_{T,i}$. In both cases $t \notin N_{T,i} - \bigcup_{j \neq i} N_{T,j}$. Because i is a leaf of G, the subnet $\bigcup_{j \neq i} N_{T,j}$ of N is connected and so even strongly-connected. This fact implies, that the net $N_{T,i} - \bigcup_{j \neq i} N_{T,j}$ contains a CP-subnet $\hat{N} \subset N$, cf. [1], Proof of Prop. 7.11. By construction $t \in N - \hat{N}$. □

2. Blocking markings

This chapter introduces the concept of blocking markings. It presents our simplified proof of the blocking theorem for bounded and live free-choice systems in Theorem 2.4.

Definition 2.1 *(Blocking marking).* A *blocking marking μ_{block} associated to a cluster c* in a Petri net is a reachable marking μ_{block} which enables every transition from c, but no other transition of the net.

If the cluster c contains only one transition t, then we talk also about a blocking marking associated to t.

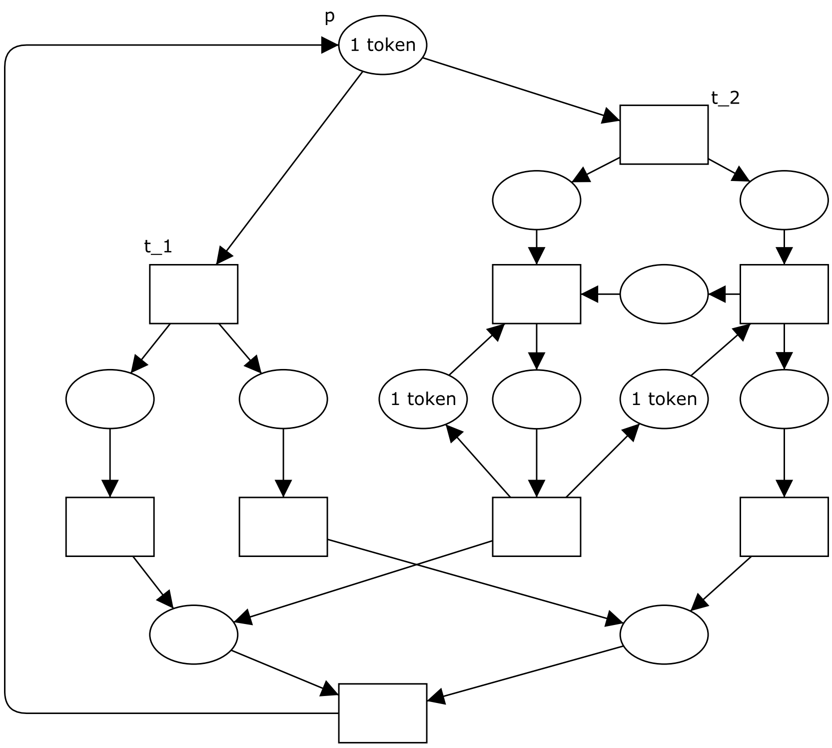

Fig. 1. Blocking marking of a cluster in a safe and live free-choice system.

Fig. 1 shows a safe and live free-choice system (N, μ_{block}) with μ_{block} a blocking marking associated to the cluster $c = \{p, t_1, t_2\}$. Fig. 2 shows a CP-subnet $\hat{N}$ of the free-choice net N from Fig. 1 together with the marking $\hat{\mu}_{block} := \mu_{block}|\hat{N}$ of $\hat{N}$ to which μ_{block} restricts. Note that $\hat{\mu}_{block}$ is a blocking marking of $(\hat{N}, \hat{\mu}_{block})$ associated to the way-in transition $t_2 \in \hat{N}$.

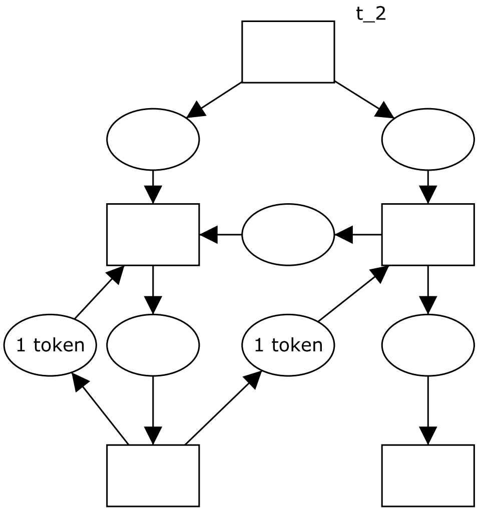

Fig. 2. CP-subnet with induced blocking marking.

Remark 2.2 collects two results about CP-subnets which serve to prove Theorem 2.4. The second result and its proof hold analogously for blocking markings in bounded and live T-systems $(\hat{N}, \hat{\mu})$.

Remark 2.2 *(CP-subnet and induced blocking marking).* Consider a well-formed free-choice net N and a CP-subnet $\hat{N} \subset N$ with way-in transition $t_{in} \in \hat{N}$, set $\bar{N} := N - \hat{N}$. Let μ be a marking of N with (N, μ) bounded and live such that the restriction $\hat{\mu} := \mu|\hat{N}$ is a blocking marking in $(\hat{N}, \hat{\mu})$ associated to t_{in}.

(i) The restriction $(\bar{N}, \mu|\bar{N})$ is a bounded and live free-choice system [1, Prop. 7.8].

(ii) For every transition $t \in \hat{N}$ there exists a path from t_{in} to t inside $\hat{N}$ which is token-free at $\hat{\mu}$ [1, Lemma 9.1].

In the proof of the Blocking Theorem 2.4 we shall show that two blocking markings of a certain free-choice system coincide. The following Lemma 2.3 prepares the proof. It derives a uniqueness result about certain markings in T-subnets of a free-choice net. The lemma will be applied to blocking markings in marked CP-subnets and in T-systems. The proof of the lemma uses ideas of Genrich and Thiagarajan from [3, Theor. 1.15], and follows the proof of [1, Lemma 9.2].

Lemma 2.3 *(Uniqueness of markings in T-subnets). Consider a strongly-connected net N, a transition-bordered T-subnet $N_T \subseteq N$ with a distinguished transition $t_0 \in N_T$ and two markings μ_k, $k = 1, 2$, of N, which agree on all S-invariants of N. Set $\mu_{k,T} := \mu_k|N_T$ and assume: For each transition $t \in N_T$ and for each index $k = 1, 2$ there exists a path from t_0 to t inside N_T, which is token-free at $\mu_{k,T}$. Then $\mu_{1,T} = \mu_{2,T}$.*

Proof. We proceed indirectly and assume w.l.o.g. that there exists a place $p_0 \in N_T$ with $\mu_{1,T}(p_0) < \mu_{2,T}(p_0)$, in particular $0 < \mu_{2,T}(p_0)$. Because N is strongly-connected and $N_T \subseteq N$ a transition-bordered T-subnet, there exists a unique pre-transition $t_{pre} \in pre(p_0) \cap N_T$ and a unique post-transition $t_{post} \in post(p_0) \cap N_T$. We select an elementary path α from t_0 to t_{pre} inside N_T, which is token-free at $\mu_{1,T}$, and denote by $\alpha_1 := \alpha \cdot \beta$ the concatenation with the path $\beta = (t_{pre}, p_0, t_{post})$. In addition, we select an elementary path α_2 from t_0 to t_{post} inside N_T, which is token-free at $\mu_{2,T}$. We have $p_0 \in \alpha_1 - \alpha_2$, because $0 < \mu_{2,T}(p_0)$. With respect to the ambient net $N = (P, T, F)$ we define

$$\lambda : P \to \mathbf{Z}, \quad \lambda(p) := \begin{cases} 1, & p \in \alpha_1 - \alpha_2, \\ -1, & p \in \alpha_2 - \alpha_1, \\ 0, & \text{else}. \end{cases}$$

That λ is an S-invariant of N is easy to see; note that one needs to verify the equation

$$\sum_{p \in pre(t)} \lambda(p) = \sum_{p \in post(t)} \lambda(p)$$

only for transitions $t \in \alpha_1 \cup \alpha_2$, because $\lambda(p) = 0$ for all $p \in P - (\alpha_1 \cup \alpha_2)$. Since λ is an S-invariant, we have by assumption that $\langle \lambda, \mu_1 \rangle = \langle \lambda, \mu_2 \rangle$. On the other hand:

$$\langle \lambda, \mu_1 \rangle = \sum_{p \in \alpha_1 \cup \alpha_2} \lambda(p) \cdot \mu_{1,T}(p) = \sum_{p \in \alpha_2 - \alpha_1} \lambda(p) \cdot \mu_{1,T}(p) = - \sum_{p \in \alpha_2 - \alpha_1} \mu_{1,T}(p) \leqslant 0,$$

but

$$\langle \lambda, \mu_2 \rangle = \sum_{p \in \alpha_1 \cup \alpha_2} \lambda(p) \cdot \mu_{2,T}(p) = \sum_{p \in \alpha_1 - \alpha_2} \lambda(p) \cdot \mu_{2,T}(p) = \sum_{p \in \alpha_1 - \alpha_2} \mu_{2,T}(p) \geqslant \mu_{2,T}(p_0) > 0.$$

This contradiction proves the lemma. □

Before entering into our proof of the Blocking Theorem 2.4 we recall the key steps in the proof of Gaujal, Haar and Mairesse. Their proof of the blocking theorem in [2, Theor. 3.1] clearly distinguishes, if an occurrence sequence contains a transition from the distinguished cluster c or not. Gaujal et al. start their proof with the particular case of a T-system. Here c contains a single transition t. By a simple argument the authors obtain an occurrence sequence σ without t, such that firing σ yields a blocking marking associated to t. They deduce that the resulting blocking marking is unique, if one admits only occurrence sequences without t. Their reasoning uses the fact that firing a transition of a T-system does not disable any other transition. To prove uniqueness also for arbitrary occurrence sequences, the authors have to convert occurrence sequences, which include t, by reverse firing to occurrence sequences, which avoid t, without changing the resulting marking.

Next, the blocking theorem for T-systems serves to start an induction. The statement for the general case of a free-choice system (N, μ) is proved by induction on the numbers of T-components $N_{T,i} \subseteq N$ from a minimal covering $(N_{T,i})_{i=1,\dots,n}$ of N. For the induction step $n-1 \mapsto n$ the authors employ CP-subnets: Set $N_+ := \bigcup_{i=1,\dots,n-1} N_{T,i}$. It is well-known that each connected component of the complement $N_- := N - N_+$ is a CP-subnet $\hat{N} \subseteq N$. In addition, there exists a flushing sequence of $\hat{N}$ with respect to (N, μ), i.e. an occurrence sequence $\mu \xrightarrow{\sigma} \hat{\mu}$ with only transitions from $\hat{N}$, such that $\hat{\mu}$ enables no transition inside $\hat{N}$ different from the way-in transition of $\hat{N}$. The complement $(N - \hat{N}, \hat{\mu}|N - \hat{N})$ is a live and bounded free-choice system. After flushing successively all CP-subnets of the complement N_- one arrives at a marking μ_+ of N, such that $(N_+, \mu_+|N_+)$ is a live and bounded free-choice system. Due to the induction assumption it has a blocking marking which can be obtained by firing an occurrence sequence without transitions from c. This marking extends to a blocking marking μ_{block} in (N, μ). In the most delicate part of their proof the authors eventually show: Any reachable marking of (N, μ), which coincides with μ_{block} on all CP-subnets $\hat{N} \subseteq N$ mentioned above, can be obtained by firing or reverse firing only transitions from N_+. After employing a further inductive reasoning this statement completes the proof of the induction step.

Our proof of Theorem 2.4 simplifies the proof of Gaujal et al. in the following respect:

- From the beginning we restrict to occurrence sequences without transitions from the distinguished cluster. It is not necessary to deal with other occurrence sequences.
- For the general case of bounded and live free-choice systems the existence of blocking markings derives directly from a simple liveness argument. It is not necessary to consider T-systems first.
- To prove the uniqueness of blocking markings we construct a finite sequence of strictly decreasing free-choice subnets of N. For the construction we remove successively CP-subnets. This construction goes by induction and ends with a T-component.
  We prove that any two blocking markings coincide on all those CP-subnets and on the final T-component. For this purpose we use the uniqueness statement for marked T-nets from Lemma 2.3. During the whole proof of the uniqueness part we hold constant the blocking markings. There is no need to fire any occurrence sequence.

Theorem 2.4 *(Blocking marking). Every cluster c in a bounded and live free-choice system (N, μ) has a unique blocking marking μ_{block} associated to it. The blocking marking is a home state of (N, μ) and can be obtained from every reachable marking without firing a transition from the cluster.*

Proof. Let X denote the set of places of c.

(i) *Existence*: Because (N, μ) is live and free-choice, there exists an occurrence sequence $\mu \xrightarrow{\sigma} \mu_c$ without transitions from c, such that the restriction $\mu_c|X$ marks each place $p \in X$. We may assume μ_c maximal with respect to X in the following sense: There does not exist any occurrence sequence $\mu \xrightarrow{\sigma} \mu'$ without transitions from c with

$$\mu'(p) \geqslant \mu_c(p) \quad \text{for all } p \in X \quad \text{and} \quad \mu'(p_0) > \mu_c(p_0) \quad \text{for at least one } p_0 \in X.$$

We define a new marking with support X

$$\mu_X : P \to \mathbf{N}, \quad \mu_X(p) := \begin{cases} \mu_c(p), & p \in X, \\ 0, & p \in P - X \end{cases}$$

and the difference marking $\mu_1 := \mu_c - \mu_X$ with no tokens at places of X.

Claim: The free-choice system (N, μ_1) is not live. Otherwise there exists an occurrence sequence $\mu_1 \xrightarrow{\sigma_1} \mu_2$ such that μ_2 enables the transitions from c by marking all places $p \in X$. We may assume, that no transition from c belongs to σ_1. Therefore also the catenation $\mu \xrightarrow{\sigma_c} \mu_1 + \mu_X \xrightarrow{\sigma_1} \mu_2 + \mu_X$ is an occurrence sequence without transitions from c. But the marking $\mu_2 + \mu_X$ contradicts the maximum property of μ_c, which proves the claim.

Because the free-choice system (N, μ_1) is not live, there exists an occurrence sequence $\mu_1 \xrightarrow{\sigma_{dead}} \mu_{dead}$ with a dead marking μ_{dead} of N, cf. [1, Theor. 4.31]. Due to the maximum property of μ_c we may assume that σ_{dead} has no transitions from c. The catenation

$$\mu \xrightarrow{\sigma_c} \mu_1 + \mu_X \xrightarrow{\sigma_{dead}} \mu_{dead} + \mu_X$$

is an occurrence sequence $\sigma_{block} := \sigma_c \cdot \sigma_{dead}$ without transitions from c. Firing σ_{block} leads to a blocking marking $\mu_{block} := \mu_{dead} + \mu_X$ associated to c in (N, μ).

(ii) *Uniqueness*: Consider two blocking markings $\mu_{k,block}$, $k = 1, 2$, associated to c in (N, μ). We set $N_0 = N$ and construct by induction a finite family $(N_i)_{i=0,\dots,n}$ of nets such that:

- $N_i \subset N_{i-1}$, $i > 0$, is a proper subnet with $c_i := c \cap N_i$ a nonempty cluster of N_i.
- $(N_i, \mu|N_i)$ is a bounded and live free-choice system.
- Both restrictions $\mu_{k,block}|N_i, k = 1, 2$, agree on all S-invariants of N_i.
- In the complement $\hat{N}_i = N_{i-1} - N_i$, $i > 0$, both restrictions $\hat{\mu}_{k,i} := \mu_{k,block}|\hat{N}_i$, $k = 1, 2$, coincide.

For the induction start $i = 0$ we note that (N, μ) is a bounded and live free-choice system and that both markings $\mu_{k,block}$, $k = 1, 2$, are reachable markings of (N, μ). Therefore they agree on all S-invariants of N.

For the induction step we assume $i \geqslant 0$ and N_i already constructed. If N_i is a T-net, we set $n = i$ and stop the iteration. Otherwise, due to Lemma 1.2 there exists a CP-subnet $\hat{N}_{i+1} \subset N_i$, which avoids at least one transition from the cluster c_i. Then $N_{i+1} := N_i - \hat{N}_{i+1}$ is a subnet of N_i with $c_{i+1} := c \cap N_{i+1} = c_i \cap N_{i+1}$ a nonempty cluster of N_{i+1}. The restriction $(N_{i+1}, \mu|N_{i+1})$ is a bounded and live free-choice system due to Remark 2.2, (i). Because $\mu_{k,block}|N_i$, $k = 1, 2$, agree on all S-invariants of N_i, also both restrictions $\mu_{k,block}|N_{i+1}$, $k = 1, 2$, agree on all S-invariants of N_{i+1}, cf. [1, Lemma 9.4]. By construction $c_i \not\subset \hat{N}_{i+1}$. Therefore each restriction $\hat{\mu}_{k,i+1}, k = 1, 2$, is a blocking marking of $(\hat{N}_{i+1}, \hat{\mu}_{k,i+1})$ associated to the way-in transition of $\hat{N}_{i+1}$. Remark 2.2, (ii) and Lemma 2.3 imply $\hat{\mu}_{1,i+1} = \hat{\mu}_{2,i+1}$, which completes the induction step.

Because the sequence $(N_i)_{i=0,\dots,n}$ of subnets is strictly decreasing, the induction terminates with a T-net N_n. For $k = 1, 2$ the two markings $\mu_{k,n} := \mu_{k,block}|N_n$ agree on all S-invariants of N_n and $(N_n, \mu_{k,n})$ are two bounded and live T-systems with blocking markings $\mu_{k,n}$ associated to the unique transition $t_0 \in c_n$. We consider N_n as T-subnet of itself and observe, that in $(N_n, \mu_{k,n})$ for each transition there exists a path which satisfies the assumption from Lemma 2.3. We obtain $\mu_{1,n} = \mu_{2,n}$, which finishes the proof for $\mu_{1,block} = \mu_{2,block}$.

(iii) *Home state*: Existence and uniqueness of the blocking marking imply that any blocking marking is a home state. □

3. Perspectives

The blocking theorem for T-system has been applied for the first time in [3] to study bipolar synchronization schemes, while the general case has been used in [2] to study routed stochastic Petri nets.

The blocking theorem singles out a subgraph Γ_{block} of the transitive closure of the case graph of a bounded and live free-choice system (N, μ_0): The vertices of Γ_{block} are the blocking markings of (N, μ_0) or equivalently the clusters of N. Two different clusters c_i, $i = 1, 2$, with associated blocking markings $\mu_{i,block}$ are connected in Γ_{block} by a directed arc iff there exists an occurrence sequence $\mu_{1,block} \xrightarrow{\sigma} \mu_{2,block}$, such that no prefix of σ yields a blocking marking of a third cluster.

How is Γ_{block} related to the full case graph, which properties of (N, μ_0) derive from Γ_{block}?

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